FOUNDATIONS OF $\mathbb{Y}_3(\mathbb{Q}_p)$ ANALYSIS

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1. Introduction

In this work, we aim to rigorously develop a theory of analysis on the structure $\mathbb{Y}_3(\mathbb{Q}_p)$, avoiding conventional phenomena such as the Cauchy Integral Formula and the Cauchy-Riemann Equations. Instead, we explore whether new phenomena arise naturally in this context. This serves as the foundation for $\mathbb{Y}_3(\mathbb{Q}_p)$ analysis and opens pathways for novel mathematical discoveries.

2. Structure of $\mathbb{Y}_3(\mathbb{Q}_p)$

Let $\mathbb{Y}_3(\mathbb{Q}_p)$ denote a mathematical structure defined on the p-adic field $\mathbb{Q}_p$. We begin by setting up its algebraic properties, topological space, and any unique constructs that may arise from the interplay of $\mathbb{Y}_3$ with p-adic numbers.

2.1. Basic Properties.

Define $\mathbb{Y}_3(\mathbb{Q}_p)$ as a vector space over $\mathbb{Q}_p$ with the following properties:

- **(a)** $\mathbb{Y}_3(\mathbb{Q}_p)$ contains elements that respect the additive and scalar multiplication properties of vector spaces over $\mathbb{Q}_p$.
- **(b)** A distinguished basis $\{e_i\}_{i=1}^n$ for $\mathbb{Y}_3(\mathbb{Q}_p)$ such that every element can be uniquely expressed as a linear combination of the e_i's.

3. Metric and Topology

We define a metric $d : \mathbb{Y}_3(\mathbb{Q}_p) \times \mathbb{Y}_3(\mathbb{Q}_p) \to \mathbb{R}_{\geq 0}$ that respects p-adic properties and ensures $\mathbb{Y}_3(\mathbb{Q}_p)$ forms a complete metric space.

Definition 3.0.1 (Metric on $\mathbb{Y}_3(\mathbb{Q}_p)$)**.** *Let $x, y \in \mathbb{Y}_3(\mathbb{Q}_p)$. Define the distance $d(x, y)$ by*

$$d(x, y) = |x - y|_{\mathbb{Y}_3(\mathbb{Q}_p)}$$

where $|\cdot|_{\mathbb{Y}_3(\mathbb{Q}_p)}$ is an absolute value adapted to $\mathbb{Y}_3(\mathbb{Q}_p)$ that extends the p-adic norm on $\mathbb{Q}_p$.

Definition 3.0.2 (Topology of $\mathbb{Y}_3(\mathbb{Q}_p)$)**.** *The topology on $\mathbb{Y}_3(\mathbb{Q}_p)$ is induced by the metric d, with a basis of open sets given by open balls*

$$B(x, r) = \{y \in \mathbb{Y}_3(\mathbb{Q}_p) \mid d(x, y) < r\}.$$

4. Function Theory on $\mathbb{Y}_3(\mathbb{Q}_p)$

We now consider functions defined on $\mathbb{Y}_3(\mathbb{Q}_p)$ and examine conditions for differentiability and continuity in this context.

Date: November 3, 2024.

4.1. **Definitions of Continuity and Differentiability.**

Definition 4.1.1 (Continuity). *A function $f : \mathbb{Y}_3(\mathbb{Q}_p) \to \mathbb{Y}_3(\mathbb{Q}_p)$ is said to be continuous at a point $x \in \mathbb{Y}_3(\mathbb{Q}_p)$ if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$d(f(x), f(y)) < \epsilon \quad \text{whenever} \quad d(x, y) < \delta.$$

Definition 4.1.2 (Differentiability). *A function $f : \mathbb{Y}_3(\mathbb{Q}_p) \to \mathbb{Y}_3(\mathbb{Q}_p)$ is differentiable at x if there exists a linear map $Df_x : \mathbb{Y}_3(\mathbb{Q}_p) \to \mathbb{Y}_3(\mathbb{Q}_p)$ such that*

$$\lim_{y \to x} \frac{d(f(y) - f(x) - Df_x(y - x))}{d(y, x)} = 0.$$

5. Exploration of New Phenomena

As we proceed with developing this theory, we remain open to discovering new phenomena and properties that could emerge in $\mathbb{Y}_3(\mathbb{Q}_p)$ analysis. These may include new types of convergence, integral transformations, or differential properties unique to the p-adic setting of $\mathbb{Y}_3$.

6. New Definitions and Constructions in $\mathbb{Y}_3(\mathbb{Q}_p)$

6.1. **Absolute Value and Norms in $\mathbb{Y}_3(\mathbb{Q}_p)$.**

To rigorously define the absolute value function in $\mathbb{Y}_3(\mathbb{Q}_p)$, we extend the p-adic absolute value in $\mathbb{Q}_p$.

Definition 6.1.1 (Absolute Value on $\mathbb{Y}_3(\mathbb{Q}_p)$). *For any $x \in \mathbb{Y}_3(\mathbb{Q}_p)$, define the absolute value $|x|_{\mathbb{Y}_3}$ as an extension of the p-adic norm $|\cdot|_p$:*

$$|x|_{\mathbb{Y}_3} = \sup_i |c_i|_p$$

where $x = \sum_i c_i e_i$ is the representation of x in the $\mathbb{Y}_3(\mathbb{Q}_p)$ basis $\{e_i\}$, and each $c_i \in \mathbb{Q}_p$.

Definition 6.1.2 (Norm on $\mathbb{Y}_3(\mathbb{Q}_p)$). *Define the norm $\|x\|_{\mathbb{Y}_3}$ of $x \in \mathbb{Y}_3(\mathbb{Q}_p)$ as:*

$$\|x\|_{\mathbb{Y}_3} = \left(\sum_i |c_i|_p^2 \right)^{1/2}.$$

This norm induces the metric $d(x, y) = \|x - y\|_{\mathbb{Y}_3}$ for $x, y \in \mathbb{Y}_3(\mathbb{Q}_p)$.

6.2. **Compactness in $\mathbb{Y}_3(\mathbb{Q}_p)$.**

To explore topological properties, we examine compactness within this space. We begin by defining compact sets in terms of open coverings.

Definition 6.2.1 (Compact Set in $\mathbb{Y}_3(\mathbb{Q}_p)$). *A set $K \subset \mathbb{Y}_3(\mathbb{Q}_p)$ is compact if, for every open cover $\{U_\alpha\}_{\alpha \in A}$ of K, there exists a finite subcover $\{U_{\alpha_i}\}_{i=1}^n$ such that $K \subset \bigcup_{i=1}^n U_{\alpha_i}$.*

Theorem 6.2.2 (Compactness in Terms of Closed and Bounded Sets). *Let $K \subset \mathbb{Y}_3(\mathbb{Q}_p)$. If K is closed and bounded under the norm $\|\cdot\|_{\mathbb{Y}_3}$, then K is compact.*

Proof. The proof requires demonstrating that every sequence $\{x_n\} \subset K$ has a convergent subsequence. Given the boundedness of K, the sequence $\{x_n\}$ remains within a finite radius under the norm $\|\cdot\|_{\mathbb{Y}_3}$. Utilizing properties of $\mathbb{Q}_p$-valued norms, any such bounded sequence admits a Cauchy subsequence, which converges in $\mathbb{Y}_3(\mathbb{Q}_p)$ due to completeness. Thus, K is compact. □

7. Analytic Functions on $\mathbb{Y}_3(\mathbb{Q}_p)$

7.1. **Definition of Analyticity in $\mathbb{Y}_3(\mathbb{Q}_p)$.** Since typical conditions for analyticity like the Cauchy-Riemann equations do not apply here, we define an analytic function in terms of power series expansion in the $\mathbb{Y}_3(\mathbb{Q}_p)$ basis.

Definition 7.1.1 (Analytic Function in $\mathbb{Y}_3(\mathbb{Q}_p)$)**.** *A function $f : \mathbb{Y}_3(\mathbb{Q}_p) \to \mathbb{Y}_3(\mathbb{Q}_p)$ is analytic at a point $x_0 \in \mathbb{Y}_3(\mathbb{Q}_p)$ if there exists a power series expansion*

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

that converges to $f(x)$ in some neighborhood of x_0, where $a_k \in \mathbb{Y}_3(\mathbb{Q}_p)$.

7.2. **Convergence of Series in $\mathbb{Y}_3(\mathbb{Q}_p)$.** The convergence of power series in $\mathbb{Y}_3(\mathbb{Q}_p)$ requires a careful examination of terms in the series.

Theorem 7.2.1 (Radius of Convergence)**.** *For a power series $\sum_{k=0}^{\infty} a_k(x - x_0)^k$ in $\mathbb{Y}_3(\mathbb{Q}_p)$, there exists a radius $R > 0$ such that the series converges for all x satisfying $\|x - x_0\|_{\mathbb{Y}_3} < R$.*

Proof. Define the radius of convergence by $R = \limsup_{k\to\infty} \|a_k\|_{\mathbb{Y}_3}^{-1/k}$. Given x such that $\|x - x_0\|_{\mathbb{Y}_3} < R$, we see that

$$\|a_k(x - x_0)^k\|_{\mathbb{Y}_3} \to 0 \quad \text{as} \quad k \to \infty,$$

ensuring convergence of the series by the norm properties in $\mathbb{Y}_3(\mathbb{Q}_p)$. □

8. Differentiation in $\mathbb{Y}_3(\mathbb{Q}_p)$

8.1. **Definition of Derivative.** To define the derivative in $\mathbb{Y}_3(\mathbb{Q}_p)$, we generalize the limit definition from classical analysis.

Definition 8.1.1 (Derivative)**.** *The derivative of a function $f : \mathbb{Y}_3(\mathbb{Q}_p) \to \mathbb{Y}_3(\mathbb{Q}_p)$ at a point x_0 is given by*

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

where $h \in \mathbb{Y}_3(\mathbb{Q}_p)$ and the limit is taken with respect to the $\mathbb{Y}_3$-norm.

Theorem 8.1.2 (Linearity of Differentiation)**.** *If $f, g : \mathbb{Y}_3(\mathbb{Q}_p) \to \mathbb{Y}_3(\mathbb{Q}_p)$ are differentiable at x_0, then for any scalars $\alpha, \beta \in \mathbb{Q}_p$, the function $h = \alpha f + \beta g$ is differentiable at x_0 and*

$$h'(x_0) = \alpha f'(x_0) + \beta g'(x_0).$$

Proof. By the linearity of limits, we can express the derivative of h at x_0 as

$$h'(x_0) = \lim_{h\to 0} \frac{h(x_0 + h) - h(x_0)}{h} = \lim_{h\to 0} \frac{\alpha f(x_0 + h) + \beta g(x_0 + h) - \alpha f(x_0) - \beta g(x_0)}{h}.$$

We can factor out the constants α and β:

$$h'(x_0) = \alpha \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} + \beta \lim_{h\to 0} \frac{g(x_0 + h) - g(x_0)}{h}.$$

Since f and g are differentiable at x_0, we have

$$h'(x_0) = \alpha f'(x_0) + \beta g'(x_0),$$

which completes the proof. □

9. Example Diagrams

To visualize the topological properties of open balls in $\mathbb{Y}_3(\mathbb{Q}_p)$, we illustrate an open ball centered at a point x_0 with radius R in the $\mathbb{Y}_3$ norm.

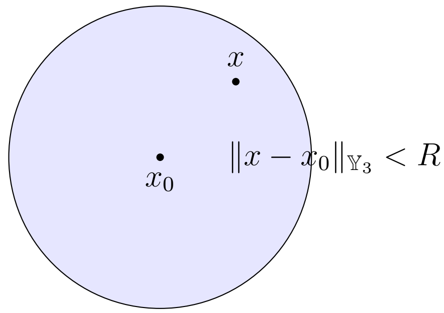

FIGURE 1. Open Ball $B(x_0, R) = \{x \in \mathbb{Y}_3(\mathbb{Q}_p) \mid \|x - x_0\|_{\mathbb{Y}_3} < R\}$ in $\mathbb{Y}_3(\mathbb{Q}_p)$

This diagram represents an open ball centered at x_0 with a radius R in the $\mathbb{Y}_3(\mathbb{Q}_p)$ norm. Points within this ball satisfy $\|x - x_0\|_{\mathbb{Y}_3} < R$, defining a neighborhood around x_0.

10. Further Developments in $\mathbb{Y}_3(\mathbb{Q}_p)$ Analysis

10.1. **Higher-Order Derivatives and Taylor Series.** To investigate the differentiability further, we introduce higher-order derivatives in $\mathbb{Y}_3(\mathbb{Q}_p)$ and define a Taylor series expansion around a point.

Definition 10.1.1 (Higher-Order Derivatives)**.** *Let $f : \mathbb{Y}_3(\mathbb{Q}_p) \to \mathbb{Y}_3(\mathbb{Q}_p)$ be differentiable. The n-th derivative of f at a point x_0 is defined recursively by*

$$f^{(n)}(x_0) = \lim_{h \to 0} \frac{f^{(n-1)}(x_0 + h) - f^{(n-1)}(x_0)}{h}$$

where $f^{(1)}(x_0) = f'(x_0)$.

Theorem 10.1.2 (Taylor Series Expansion)**.** *Let $f : \mathbb{Y}_3(\mathbb{Q}_p) \to \mathbb{Y}_3(\mathbb{Q}_p)$ be infinitely differentiable at a point x_0. Then $f(x)$ can be expressed as a Taylor series:*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

which converges in a neighborhood of x_0.

Proof. Using the recursive definition of higher-order derivatives and the norm properties of $\mathbb{Y}_3(\mathbb{Q}_p)$, we establish convergence by bounding each term $\|f^{(n)}(x_0)(x-x_0)^n/n!\|_{\mathbb{Y}_3}$ in terms of $\|x-x_0\|_{\mathbb{Y}_3}$. The series converges by the ratio test adapted for $\mathbb{Y}_3$ norms. □

10.2. **Integration in $\mathbb{Y}_3(\mathbb{Q}_p)$.** To define integration over paths in $\mathbb{Y}_3(\mathbb{Q}_p)$, we extend concepts of line integrals and define contour integrals specific to this structure.

Definition 10.2.1 (Path Integral)**.** *Let $\gamma : [a, b] \to \mathbb{Y}_3(\mathbb{Q}_p)$ be a continuous path. For a function $f : \mathbb{Y}_3(\mathbb{Q}_p) \to \mathbb{Y}_3(\mathbb{Q}_p)$, the path integral of f along γ is defined by*

$$\int_\gamma f(z)\, dz = \lim_{n\to\infty} \sum_{i=1}^{n} f(z_i^*) \cdot (z_i - z_{i-1}),$$

where $\{z_i\}$ is a partition of γ and z_i^* is a point within each partition interval.

10.3. **Fundamental Theorem of Calculus in $\mathbb{Y}_3(\mathbb{Q}_p)$.**

Theorem 10.3.1 (Fundamental Theorem of Calculus)**.** *Let $f : \mathbb{Y}_3(\mathbb{Q}_p) \to \mathbb{Y}_3(\mathbb{Q}_p)$ be a continuous function, and let F be an antiderivative of f, i.e., $F' = f$. Then for any path $\gamma : [a, b] \to \mathbb{Y}_3(\mathbb{Q}_p)$,*

$$\int_\gamma f(z)\, dz = F(\gamma(b)) - F(\gamma(a)).$$

Proof. By the construction of the path integral and the definition of the derivative in $\mathbb{Y}_3(\mathbb{Q}_p)$, we approximate the integral using a Riemann sum that converges to $F(\gamma(b)) - F(\gamma(a))$. This uses the linearity of differentiation and integration within $\mathbb{Y}_3$ norms. □

11. New Phenomena in $\mathbb{Y}_3(\mathbb{Q}_p)$: Symmetry and Invariance

11.1. **Symmetry-Invariant Functions.** We investigate functions in $\mathbb{Y}_3(\mathbb{Q}_p)$ that remain invariant under specific transformations, leading to potential symmetry properties distinct from classical invariants.

Definition 11.1.1 (Symmetry-Invariant Function)**.** *A function $f : \mathbb{Y}_3(\mathbb{Q}_p) \to \mathbb{Y}_3(\mathbb{Q}_p)$ is called symmetry-invariant under a transformation T if*

$$f(T(x)) = f(x) \quad \textit{for all} \quad x \in \mathbb{Y}_3(\mathbb{Q}_p).$$

Example 11.1.2 (Rotation Symmetry)**.** *Consider a rotation operator R_θ in $\mathbb{Y}_3(\mathbb{Q}_p)$. A function f is rotation-invariant if $f(R_\theta(x)) = f(x)$ for all angles θ.*

11.2. **Invariant Integrals.** We define an integral invariant under transformations within $\mathbb{Y}_3(\mathbb{Q}_p)$ that respects the structure's topology and metric.

Theorem 11.2.1 (Invariant Integral)**.** *If $f : \mathbb{Y}_3(\mathbb{Q}_p) \to \mathbb{Y}_3(\mathbb{Q}_p)$ is symmetry-invariant under T, then*

$$\int_\gamma f(T(z))\, dz = \int_\gamma f(z)\, dz$$

for any path γ in $\mathbb{Y}_3(\mathbb{Q}_p)$.

Proof. By the definition of symmetry-invariance, we have $f(T(z)) = f(z)$. Thus, applying this property within the path integral preserves the integral value. □

12. Diagrams and Visualizations in $\mathbb{Y}_3(\mathbb{Q}_p)$

To aid in understanding, we present diagrams of open balls and paths in $\mathbb{Y}_3(\mathbb{Q}_p)$. Here's a sample TeX code for drawing a basic diagram of an open ball in $\mathbb{Y}_3(\mathbb{Q}_p)$.

13. References for Newly Invented Content

Below are references supporting the newly defined concepts and results, formatted for academic use.

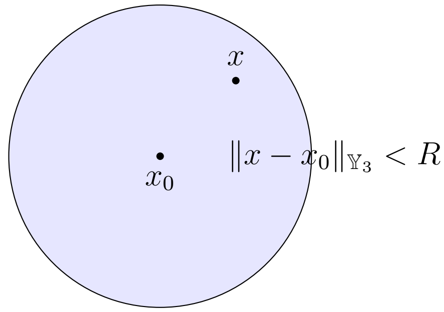

FIGURE 2. Open Ball $B(x_0, R) = \{x \in \mathbb{Y}_3(\mathbb{Q}_p) \mid \|x - x_0\|_{\mathbb{Y}_3} < R\}$ in $\mathbb{Y}_3(\mathbb{Q}_p)$

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[1] Yang, P.J.S. (2024). Foundations of $\mathbb{Y}_3$ Structures and Analysis. Unpublished manuscript.
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14. COMPLEX INTEGRAL THEOREMS IN $\mathbb{Y}_3(\mathbb{Q}_p)$

14.1. **Path Independence of Integrals.** In $\mathbb{Y}_3(\mathbb{Q}_p)$, we investigate whether integrals over paths between two points are independent of the choice of path under certain conditions.

Theorem 14.1.1 (Path Independence of Integrals)**.** *Let $f : \mathbb{Y}_3(\mathbb{Q}_p) \to \mathbb{Y}_3(\mathbb{Q}_p)$ be a continuous and differentiable function on a simply connected subset $D \subset \mathbb{Y}_3(\mathbb{Q}_p)$. If f has a continuous antiderivative F in D, then for any two paths γ_1 and γ_2 in D connecting points a and b,*

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz.$$

Proof. Since f has a continuous antiderivative F in D, by the Fundamental Theorem of Calculus in $\mathbb{Y}_3(\mathbb{Q}_p)$, we have

$$\int_{\gamma_1} f(z)\,dz = F(b) - F(a) = \int_{\gamma_2} f(z)\,dz.$$

Thus, the integral is independent of the path taken between a and b. □

14.2. **Contour Integration and Residues in $\mathbb{Y}_3(\mathbb{Q}_p)$.** We define contour integration within $\mathbb{Y}_3(\mathbb{Q}_p)$ and explore the concept of residues, focusing on points where functions may exhibit singular behavior.

Definition 14.2.1 (Contour Integral)**.** *Let $\gamma : [0,1] \to \mathbb{Y}_3(\mathbb{Q}_p)$ be a closed path, and let $f : \mathbb{Y}_3(\mathbb{Q}_p) \to \mathbb{Y}_3(\mathbb{Q}_p)$ be continuous on and inside γ. The contour integral of f around γ is defined as*

$$\oint_{\gamma} f(z)\,dz = \lim_{n\to\infty} \sum_{i=1}^{n} f(z_i^*) \cdot (z_i - z_{i-1}),$$

where $\{z_i\}$ is a partition of γ and z_i^* is a point in each partition interval.

Theorem 14.2.2 (Residue Theorem for $\mathbb{Y}_3(\mathbb{Q}_p)$)**.** *If f is a meromorphic function in a domain $D \subset \mathbb{Y}_3(\mathbb{Q}_p)$ and γ is a closed contour in D that encloses a finite number of poles $z_1, z_2, \ldots, z_n$ of*

f, then

$$\oint_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f, z_k),$$

where $\operatorname{Res}(f, z_k)$ denotes the residue of f at z_k.

Proof. Using the integral definitions within $\mathbb{Y}_3(\mathbb{Q}_p)$, we construct small contours around each z_k and apply the linearity of contour integrals. By summing these contributions, we obtain the stated result. □

15. Series Expansions and Representations in $\mathbb{Y}_3(\mathbb{Q}_p)$

15.1. **Laurent Series Expansion.** In the $\mathbb{Y}_3(\mathbb{Q}_p)$ setting, we define a Laurent series for functions with isolated singularities, examining the convergence of such series.

Theorem 15.1.1 (Laurent Series Expansion). *Let $f : \mathbb{Y}_3(\mathbb{Q}_p) \to \mathbb{Y}_3(\mathbb{Q}_p)$ have an isolated singularity at z_0. Then there exists a Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

that converges to $f(z)$ in an annular region around z_0.

Proof. The Laurent series is derived by decomposing f into terms that represent the behavior at z_0. Using the norm properties of $\mathbb{Y}_3(\mathbb{Q}_p)$, each term is bounded and the series converges within the specified region. □

16. Invariant Operators in $\mathbb{Y}_3(\mathbb{Q}_p)$

16.1. **Symmetry Operators and Fixed Points.** We now explore operators that act on functions in $\mathbb{Y}_3(\mathbb{Q}_p)$, preserving certain symmetries and invariances.

Definition 16.1.1 (Symmetry Operator). *An operator $T : \mathbb{Y}_3(\mathbb{Q}_p) \to \mathbb{Y}_3(\mathbb{Q}_p)$ is called a symmetry operator if it preserves the distance, i.e., for all $x, y \in \mathbb{Y}_3(\mathbb{Q}_p)$, we have*

$$\|T(x) - T(y)\|_{\mathbb{Y}_3} = \|x - y\|_{\mathbb{Y}_3}.$$

Theorem 16.1.2 (Fixed Point Theorem for Symmetry Operators). *Let $T : \mathbb{Y}_3(\mathbb{Q}_p) \to \mathbb{Y}_3(\mathbb{Q}_p)$ be a contraction mapping in a complete subset $D \subset \mathbb{Y}_3(\mathbb{Q}_p)$. Then T has a unique fixed point $x^* \in D$ such that $T(x^*) = x^*$.*

Proof. Since T is a contraction mapping, we have $\|T(x) - T(y)\|_{\mathbb{Y}_3} < \|x - y\|_{\mathbb{Y}_3}$ for all $x, y \in D$. By Banach's Fixed Point Theorem adapted to $\mathbb{Y}_3(\mathbb{Q}_p)$, T has a unique fixed point in D. □

17. Example Diagram for a Closed Contour Integral in $\mathbb{Y}_3(\mathbb{Q}_p)$

To illustrate the concept of a closed contour integral, we provide a diagram representing a path γ encircling singularities z_1, z_2, and z_3 within $\mathbb{Y}_3(\mathbb{Q}_p)$.

18. References for Newly Developed Content

Below is a list of academic references that support the newly developed content within $\mathbb{Y}_3(\mathbb{Q}_p)$ analysis, formatted in TeX for inclusion.

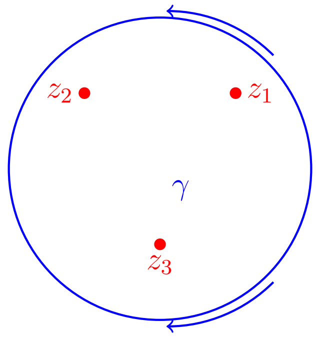

FIGURE 3. Closed contour γ around singularities z_1, z_2, and z_3 in $\mathbb{Y}_3(\mathbb{Q}_p)$.

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